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# THE POISSON SET OF CRACKS IN AN ELASTIC CONTINUOUS MEDIUM*

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The method of effective (self-consistent) field is used for solving the problem of a random set of interacting cracks in an elastic medium. Construction of the first moment of solution is shown on the example of a medium containing Poisson set of plane elliptic cracks. Effective elastic constants of a medium with cracks are determined and the results obtained in the plane case are compared with published experimental data.

In a number of publications on the mechanics of inhomogeneous elastic media /1—3/ self-consistent solutions were derived on the assumption that each typical inhomogeneity (a polycrystal grain, inclusion in the composite, or crack) behaves as an isolated one in an otherwise homogeneous medium whose properties are the same as the effective properties of the whole medium, with the field containing any inhomogeneity assumed equal to the external field. This modification of the self-consistency method is sometimes called the method of effective medium.

Another scheme of solution derivation may be proposed in which every typical particle (inhomogeneity) is considered as isolated in the base medium (matrix) of known properties and the presence of surrounding particles taken into account by the effective external field containing that particle. A similar scheme, called below the method of the effective field, is developed here. (Such method was used in /4/ for investigating an elastic composite medium).

1. Let us consider a homogeneous three-dimensional elastic body containing a set of arbitrarily situated cracks— slits along smooth oriented surfaces whose edges are for simplicity considered to be free of external loads.

In a number of cases of practical interest crack dimensions and the distance between them are considerably smaller than the dimensions of the body and the characteristic scale of the external field. Because of this, an unbounded elastic medium with cracks in a constant external stress field induced by loads applied at infinity is considered below.

As shown in /5/, an adequate model of a crack is a surface which carried dislocation density moments $n_\alpha(x)b_\beta(x)$, where $n(x)$ is a normal to the crack surface $\Omega$ and $b(x)$ is the vector of the displacement jump at transition through $\Omega$ ($x(x_1, x_2, x_3)$ is a point of the medium)

Any arbitrary set of cracks in a homogeneous medium may be considered as the distribution of dislocation density moments $m(x)$ of the form

$$m(x)=\sum_k m^{(k)}(x),\quad m_{\alpha\beta}^{(k)}(x)=n_\alpha^{(k)}(x)\,b_\beta^{(k)}(x)\,\Omega_k(x) \tag{1.1}$$

where $n^{(k)}(x)$ is the normal to the surface of the $k$-th crack $\Omega_k$ and $\Omega_k(x)$ is the delta function concentrated on $\Omega_k$ (for the definition of $\Omega_k(x)$ see /6/, where this function is denoted by $\delta(\Omega_k)$).

Vectors $b^{(k)}(x)$ are jumps of displacement fields on $\Omega_k$ that are to be determined by solving the elastic problem for a medium with cracks.

When $b^{(k)}(x)$ are known, then, using the continuum theory of dislocation /6/, the stress $\sigma(x)$ and strain $\varepsilon(x)$ fields in a medium with cracks can be determined using formulas

$$\sigma^{\alpha\beta}(x)=\sigma_0^{\alpha\beta}+\int S^{\alpha\beta\lambda\mu}(x-x')\,m_{\lambda\mu}(x')\,dx',\quad \varepsilon_{\alpha\beta}(x)=\varepsilon_{0\alpha\beta}+\int K_{\alpha\beta\lambda\mu}(x-x')\,c^{\lambda\mu\nu\rho}m_{\nu\rho}(x')\,dx' \tag{1.2}$$

where $\sigma_0$ and $\varepsilon_0$ are, respectively,the stress and strain fields, $\sigma_0^{\alpha\beta}=c^{\alpha\beta\lambda\mu}\varepsilon_{0\lambda\mu}$, $c^{\alpha\beta\lambda\mu}$ is the tensor of the homogeneous medium elasticity moduli, and $m(x)$ is of the form (1.1).

The kernels of integral operators $S$ and $\mathbf{K}$ in (1.2) are expressed in terms of second derivatives of Green's function $G(x)$ of the homogeneous medium

$$S^{\alpha\beta\lambda\mu}(x)=c^{\alpha\beta\nu\rho}K_{\nu\rho\tau\sigma}(x)\,c^{\tau\sigma\lambda\mu}-c^{\alpha\beta\lambda\mu}\delta(x),\quad K_{\alpha\beta\lambda\mu}(x)=-[\nabla_\alpha\nabla_\lambda G_{\beta\mu}(x)]_{(\alpha\beta)(\lambda\mu)} \tag{1.3}$$

where $\delta(x)$ is the Dirac delta function and $\nabla$ is the gradient operator in the three-dimensional Euclidean space $\mathbf{R}^3$.

It follows from (1.3) that $S(x)$ and $K(x)$ are generalized homogeneous functions of power $-3$.

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Let us consider the properties of operators $\mathbf{S}$ and $\mathbf{K}$ in (1.2). They are convolution operators with generalized functions $S(x)$ and $K(x)$. In functions of class $S(\mathbf{R}^3)$ (as $|x| \to \infty$ they decrease more rapidly than any power of $|x|$) $S$ and $K$ admit the regular representations /7/

$$\int S(x-x')\varphi(x')\,dx' = \int S(x-x')\varphi(x')\,dx' + D\varphi(x) \tag{1.4}$$

$$\int K(x-x')c\varphi(x')\,dx' = \int K(x-x')c\varphi(x')\,dx' + Ac\varphi(x) \tag{1.5}$$

where $(\varphi \in S(\mathbf{R}^3))$ and the integrals on the right exist and are taken in the sense of Cauchy's principal value, the constant tensors $D$ and $A$ are expressed in terms of Fourier transforms $S(k)$ and $K(k)$ and of functions $S(x)$ and $K(x)$ by formulas

$$D = \frac{1}{4\pi}\int_{\Omega_1} S(k)\,d\Omega, \quad A = \frac{1}{4\pi}\int_{\Omega_1} K(k)\,d\Omega \tag{1.6}$$

where $\Omega_1$ is the surface of a unit sphere in the $k$-th space of Fourier transforms.

Let us now determine the effect of the action of operators $\mathbf{S}$ and $\mathbf{K}$ on the constant $m_0$. The respective integrals formally diverge at zero and infinity. Note that the integrals

$$\int S(x-x')m_0\,dx', \quad \int K(x-x')cm_0\,dx'$$

have the meaning of the fields of internal stresses and strains, respectively, in a homogeneous medium containing dislocation moments of constant density $m_0$. If the medium deformation is not constrained at infinity, such distribution of dislocations does not result in the appearance of internal stresses, but induces an additional constant ("plastic") deformation of the medium of magnitude equal to $m_0$. Consequently in this case

$$\int S(x-x')m_0\,dx' = 0, \quad \int K(x-x')cm_0\,dx' = m_0 \tag{1.7}$$

However, if the medium deformation is constrained at infinity, the result is obviously different. There is no unambiguous natural definition of operators $\mathbf{S}$ an $\mathbf{K}$ constants, and in specific problems the respective integrals are determined by their particular meaning.

On the assumption that deformations are not constrained at infinity, we use below formulas (1.7) and consider only homogeneous random crack sets in space. The density $m(x)$ in (1.2) is also a generalized homogeneous random field.

It can be shown that formulas (1.4)–(1.7) are sufficient for determining operators $\mathbf{S}$ and $\mathbf{K}$ on smooth homogeneous random functions. In the model of the random field $m(x)$ these operators are continuously extended.

2. Let us take one of the typical models of random crack sets homogeneous in space, and consider an arbitrary crack numbered $i$. If functions $m^{(k)}(x)$ defined by formula (1.1) and appropriate to the considered model are known, the field $\bar{\sigma}_i(x)$ that contains the $i$-th crack is by virtue of the first of formulas (1.2) of the form

$$\bar{\sigma}_i(x) = \sigma_0 + \sum_{k\neq i}\int S(x-x')m^{(k)}(x')\,dx', \quad x \in \Omega_i \tag{2.1}$$

The field $\bar{\sigma}_i(x)$ defined by this formula at points of surface $\Omega_i$ can be taken as the external field of the $i$-th crack in which it behaves an an isolated one.

Let us assume that the solution of the elastic problem for a crack isolated in an arbitrary external field $\bar{\sigma}_k(x)$ is known. This implies that function $m^{(k)}(x, \bar{\sigma}_k)$ is known in explicit form. The system of equations that satisfies fields $\bar{\sigma}_i(x)$ for each of the interacting cracks then follows from (2.1) and is of the form

$$\bar{\sigma}_i(x) = \sigma_0 + \sum_{k\neq i}\int S(x-x')m^{(k)}(x, \bar{\sigma}_k)\,dx', \quad x \in \Omega_i, \quad i = 1, 2, \ldots \tag{2.2}$$

When the solution of this system is known, it is possible to obtain from (1.2) the stresses and strains in a medium with cracks. Fields $\bar{\sigma}_i(x)$ may be considered as the basic unknowns of the problem.

If the crack set is random, $\bar{\sigma}_i(x)$ are random functions. The construction of statistical moments of fields $\bar{\sigma}_i(x)$ reduces to solving the problem of interaction between many cracks, which involves virtually unsurmountable difficulties. To make the problem more tractable it is necessary to introduce simplifying assumptions about the structure of random fields $\bar{\sigma}_i(x)$.

We shall assume the field $\bar{\sigma}_i(x)$ to be virtually constant in region $\Omega_i$ but, generally, different for individual cracks (hypothesis $H_1$) and consider all cracks to be plane and of elliptic form.

The solution of the problem of an isolated elliptic crack in a constant external stress field $\bar\sigma_k$ shows that the quantities $m^{(k)}(x, \bar\sigma_k)$ in (2.2) are determined by formulas

$$m_{\alpha\beta}^{(k)}(x,\bar\sigma_k) = P_{\alpha\beta\lambda\mu}^{(k)}(x)\bar\sigma_k^{\lambda\mu}\Omega_k(x), \quad P^{(k)}(x) = P^{(k)}h_k(x) \tag{2.3}$$

where the scalar function $h_k(x)$ determined at points of surface $\Omega_k$ in the system of coordinates $x_1, x_2$ attached to the principal axes of the crack is of the form

$$h_k(x_1, x_2) = \frac{a_k^2}{b_k}\left(1 - \frac{x_1^2}{a_k^2} - \frac{x_2^2}{b_k^2}\right)^{1/2}$$

where $a_k$ and $b_k$ are semiaxes of the ellipse $\Omega_k$, $a_k \geqslant b_k$.

The constant tensor $P^{(k)}$ in (2.3) is of the form

$$P_{\alpha\beta\lambda\mu}^{(k)} = n_{(\alpha}^{(k)}T_{\beta)(\lambda}^{(k)}n_{\mu)}^{(k)} \tag{2.4}$$

where for an isotropic medium $T_{\alpha\beta}^{(k)}$ is of the form (no summation with respect to $\alpha$!)

$$T_{\alpha\beta}^{(k)} = \frac{2a_k^2}{b_k}\frac{1-\nu}{\mu}d_\alpha^{-1}\delta_{\alpha\beta}$$

$$d_1 = c_1 + \nu(c_2 - 2c_1), \quad d_2 = c_1 + \nu(c_3 - 2c_1), \quad d_3 = c_1/2$$

$$c_3 = -c_2 + 3c_1$$

$$c_1 = E(\alpha)/1 - \alpha^2, \quad c_2 = c_1 - (E(\alpha) - K(\alpha))/\alpha^2,$$

$$\alpha = 1 - (b_k/a_k)^2$$

where $\mu$ is the shear modulus, $\nu$ is the Poisson's ratio of the medium, $\delta_{\alpha\beta}$ is the Kronecker delta, and $E(\alpha)$ and $K(\alpha)$ are complete elliptic integrals of the first and second kind.

We introduce the following notation: $\Omega$ for a set of crack surfaces in space, and $\Omega(x)$ for the delta function concentrated on $\Omega$. We fix some point $x_0 \in \Omega$ and define region $\Omega_{x_0}$ by the formula

$$\Omega_{x_0} = \bigcup_{i\neq j}\Omega_i, \quad x_0 \in \Omega_j$$

We denote the delta function concentrated on $\Omega_{x_0}$ by $\Omega(x_0; x)$.

Let $P(x)$ be an arbitrary smooth tensor field coinciding on surfaces $\Omega_k$ with $P^{(k)}(x)$ (see (2.3)). We introduce the field $\bar\sigma(x)$ defined in region $\Omega$ by the equation

$$\bar\sigma(x) = \sigma_0 + \int S(x - x')P(x')\bar\sigma(x')\Omega(x, x')dx', \quad x \in \Omega \tag{2.5}$$

The comparison with (2.2) and (2.3) shows that when the $H_1$ hypothesis holds, the field $\bar\sigma(x)$ coincides in regions $\Omega_k$ with $\bar\sigma_k$. We call $\bar\sigma(x)$ the effective field for a given set of cracks.

In the first approximation it is possible to assume that the effective field $\bar\sigma(x)$ is constant and the same for all cracks (a similar idea was used in /9,10/ for solving the problem of wave propagation in a medium with random inhomogeneities). This assumption is evidently valid when the field of each crack weakly depends on the configuration of the set of surfaces $\Omega_k$ and is determined by the average combined field of all interacting cracks. Then, by averaging over the ensemble of models of random crack sets under the condition that $x \in \Omega$, we obtain

$$\bar\sigma = \sigma_0 + \int S(x - x') < P(x')\Omega(x; x') \mid x > dx'\bar\sigma \tag{2.6}$$

where it is taken into consideration that $\bar\sigma = \text{const}$ for $x \in \Omega$, and $\langle\cdot \mid x\rangle$ implies averaging under the condition that $x \in \Omega$.

The problem has been, thus, reduced to the construction and calculation of the integral in (2.6). Since for a uniform field of cracks the average of the integrand which is a function of the remainder $x - x'$, the integral is some constant. Solving (2.6) for the tensor $\bar\sigma$ we obtain

$$\bar\sigma^{\alpha\beta} = \Lambda_{\lambda\mu}^{\alpha\beta}\sigma_0^{\lambda\mu} \tag{2.7}$$

$$\Lambda = [I - \int S(x - x') < P(x')\Omega(x; x') \mid x > dx']^{-1} \tag{2.8}$$

where $I$ is a unit tensor of the fourth rank.

Let us determine the mathematical expectations of the stress and strain fields in a medium with cracks. Substituting $m^{(k)}(x, \bar\sigma)$ of form (2.3) into (1.1) and, then, its result into (1.2), and averaging the obtained expressions for $\sigma(x)$ and $\varepsilon(x)$ over the ensemble of crack sets, we obtain

$$\langle \sigma(x)\rangle = \sigma_0 + \int S(x-x')\langle P(x')\Omega(x')\rangle dx'\overline{\sigma}, \quad \langle \varepsilon(x)\rangle = \varepsilon_0 + \int K(x-x')c\langle P(x')\Omega(x')\rangle dx'\overline{\sigma} \tag{2.9}$$

For a set of cracks uniform in space the average $\langle P(x)\Omega(x)\rangle$ is a constant quantity. The action of operators S and K on constants is determined by formulas (1.7), hence from (2.9) follows that

$$\langle \sigma(x)\rangle = \sigma_0, \quad \langle \varepsilon(x)\rangle = \varepsilon_0 + \langle P(x)\Omega(x)\rangle \overline{\sigma} \tag{2.10}$$

We introduce the effective compliance tensor $B_*$ for a medium with cracks by the natural relation

$$\langle \varepsilon\rangle = B_*\langle \sigma\rangle \tag{2.11}$$

which together with (2.10) and (2.7) yields the equality

$$B_* = B + \langle P(x)\Omega(x)\rangle \Lambda, \quad B = c^{-1} \tag{2.12}$$

where $\Lambda$ is of the form (2.8).

Let us compare the method of effective field for determining the tensor of effective elastic constants with the method of effective medium /1– 3/ based on the assumption that each crack behaves as an isolated one in a homogeneous medium whose properties are the same as the effective properties of the whole medium with cracks. The external field of any crack is assumed equal to $\sigma_0$.

On these assumptions formulas (1.2) for $\sigma(x)$ and $\varepsilon(x)$ in the case of a medium with elliptic cracks assume the form

$$\sigma(x) = \sigma_0 + \int S(x-x')P_*(x')\Omega(x')dx'\sigma_0, \quad \varepsilon(x) = \varepsilon_0 + \int K(x-x')cP_*(x')\Omega(x')dx'\sigma_0 \tag{2.13}$$

$$P_*(x)\Omega(x) = \sum_k P_*^{(k)}(x)\Omega_k(x)$$

where functions $P_*^{(k)}(x)$, although of a form similar to that shown in (2.3), are determined by solving the problem of an isolated elliptic crack in a homogeneous medium with the elastic pliability tensor $B_*$. The equation for $B_*$ is obtained from the self-consistency condition which conforms to (2.11). Averaging expressions (2.13) for $\sigma(x)$ and $\varepsilon(x)$, then acting by operators S and K on the constant $\langle P_*(x)\Omega(x)\rangle$, and substituting the result into (2.11), we obtain

$$B_* = B + \langle P_*(x)\Omega(x)\rangle \tag{2.14}$$

This relation which links the components of tensor $B_*$ is to be considered as the equation of the effective elastic constants of a medium with cracks (these constants appear in the right-hand side of this equation because of the tensor $P_*(x)$ whose explicit expression is assumed known).

3. Let us analyze some specific models of random crack sets in space. The ergodicity of the considered random functions assumed below, enables us to substitute averages over volume for averages over ensembles of models for a fixed typical model. Thus, for example,

$$\langle P(x)\Omega(x)\rangle = \lim_{v\to\infty}\frac{1}{v}\int_V \sum_{k=1}^{N} P^{(k)}(x)\Omega_k(x)dx = \lim_{v\to\infty}\frac{1}{v}\sum_{k=1}^{N}\frac{2\pi}{3}a_k^3 P^{(k)} \tag{3.1}$$

where $V$ is a region of volume $v$ in $R^3$, which at the limit occupies the whole space, $N$ is the number of cracks contained in $V$, and $P^{(k)}(x)$ and $P^{(k)}$ are of the form appearing in (2.3) and (2.4).

In the ensemble of models of the random field $P(x)\Omega(x)$ tensors $a_k^3 P^{(k)}$ are random quantities with one and the same distribution function for all $k$. Averaging once more both sides of (3.1) over the ensemble of models, we obtain

$$\langle P(x)\Omega(x)\rangle = \lim_{v\to\infty}\frac{N}{v}\left\langle \frac{2\pi}{3}a^3 P(a,b)\right\rangle = \frac{2\pi}{3v_0}\langle a^3 P(a,b)\rangle \tag{3.2}$$

where $v_0$ is the average volume of a single crack. The average random tensor $a^3P(a,b)$, where $P(a,b)$, which appears here in the right-hand side, is of the same form as $P^{(k)}$; its value is determined by the distribution of random semiaxes $a$ and $b$ and of its random orientation.

Let us consider the conditional average in (2.6) and (2.8). By the definition of the condition mean we have

$$\Psi(x-x') = \langle P(x')\Omega(x;x')\,|\,x\rangle = \frac{\langle P(x')\Omega(x;x')\Omega(x)\rangle}{\langle \Omega(x)\rangle} \tag{3.3}$$

Similarly to (3.1) we have

$$\langle \Omega(x)\rangle = \pi\langle ab\rangle / v_0, \quad \Psi(x) = \frac{v_0}{\pi\langle ab\rangle}\lim_{v\to\infty}\frac{1}{v} \times \int_V P(x-x')\,\Omega(x';x'-x)\,\Omega(x')\,dx' \tag{3.4}$$

Let us consider some examples of stochastic crack sets in space.

1°. **The Poisson set of cracks.** Let the bounded volume $V$ contain $N$ cracks whose dimensions and orientations are random quantities with known distribution functions. The coordinates of crack centers are independent random quantities uniformly distributed in $V$. Making $V$ and $N$ approach infinity so that $v/N \to v_0 < \infty$, we obtain a set of cracks which is uniform in space, which we shall call the Poisson set. By virtue of the uncorrelated position of crack centers in the Poisson set we have

$$\langle P(x')\Omega(x;x')\Omega(x)\rangle = \langle P(x')\Omega(x;x')\rangle\langle\Omega(x)\rangle \tag{3.5}$$

where similarly to (3.1)

$$\langle P(x')\,\Omega(x;x')\rangle = \lim_{v\to\infty}\frac{1}{v}\sum_{k=1}^{N}{}' \frac{2\pi}{3}a_k^3P^{(k)} = \frac{2\pi}{3v_0}\langle a^3P(a,b)\rangle \tag{3.6}$$

where the prime at the summation symbol means the omission of the term $\frac{2\pi}{3}a_i^3P^{(i)}$ for $x\in\Omega_i$, as implied by the definition of function $\Omega(x;x')$.

Thus generally $\Psi(x) = \text{const}$ (except for the set of points $x$ of zero measure). By virtue of (1.7) the integral in (2.6) and (2.8) vanishes, and formulas (2.12) for $B_*$ assumes the form

$$B_* = B + \frac{2\pi}{3v_0}\langle a^3P(a,b)\rangle \tag{3.7}$$

in which allowance is made for (3.2).

In the case of circular cracks of random radius $a$ in an isotropic medium from this formula we obtain

$$B_* = B - \frac{8}{3}\frac{1-\nu}{\mu(2-\nu)}q\left\langle\frac{1}{2}Q^1(n) - \nu Q^2(n)\right\rangle, \quad q = \frac{\langle a^3\rangle}{v_0} \tag{3.8}$$

$$Q^1_{\alpha\beta\lambda\mu} = n_\alpha n_\lambda\delta_{\beta\mu} + n_\alpha n_\mu\delta_{\beta\lambda} + n_\beta n_\lambda\delta_{\alpha\mu} + n_\beta n_\mu\delta_{\alpha\lambda}, \quad Q^2_{\alpha\beta\lambda\mu} = n_\alpha n_\beta n_\lambda n_\mu$$

in which summation is carried out with respect to random orientation of cracks $n$ (the random quantity $a$ is assumed independent of $n$).

When the distribution with respect to orientations is uniform, $B_*$ is an isotropic tensor, and the effective shear modulus $\mu_*$ and the Poisson's ratio $\nu_*$ of a medium with cracks assume the form

$$\mu_* = \mu\left[1 + \frac{32}{45}q\frac{(1-\nu)(5-\nu)}{(2-\nu)}\right]^{-1}, \quad \frac{\nu_*}{1+\nu_*} = \frac{\nu}{1+\nu}\frac{\mu_*}{\mu}\left[1 + \frac{16}{45}q\frac{1-\nu^2}{2-\nu}\right] \tag{3.9}$$

2°. **Model with restriction on crack intersection.** Let us consider a Poisson's model with one additional condition. Let the neighborhood (for instance, spherical) of each crack be such that the probability of other crack centers reaching it is small, and a short-range order obtains in the random set of cracks. Function $\Psi(x)$ defined by formulas (3.3) is zero in some neighborhood of the coordinate origin (point $x = 0$). As $|x - x'| \to \infty$ the correlation of crack positions vanishes and

$$\Psi(x-x') \to \frac{2\pi}{3v_0}\langle a^3P(a,b)\rangle = \Psi_*$$

(the quantity $\Psi_*$ is the same as $\Psi(x)$ for the Poisson set of cracks (3.6)).

When the crack concentration is not excessive, then, owing to the isotropy of the model, $\Psi(x)$, spherically symmetric $\Psi(x) = \Psi(|x|)$ and the integral in (2.6) and (2.8) is of a particularly simple form

$$\int S(x-x')\Psi(x-x')dx' = \int S(x-x')[\Psi(x-x') - \Psi_*]dx' = -D\Psi_*$$

where allowance is made for formulas (1.4) and (1.7), tensor $D$ is of the form (1.6), and the integral appearing in (1.4) in the sense of principal value vanishes owing to the spherical symmetry of $\Psi(x)$.

For $B_*$ we obtain from (2.1) an expression of the form

$$B_* = B + \frac{2\pi}{3v_0}\langle a^3P(a,b)\rangle\left[I - \frac{2\pi}{3v_0}D\langle a^3P(a,b)\rangle\right]^{-1}$$

In the case of an isotropic medium and uniform orientation distribution we obtain from this

$$B_* = B + \frac{8}{45}\frac{1-\nu}{\mu(2-\nu)}\frac{q}{(1-\beta_1)} \times \left[\frac{2\beta_2(5-\nu)-\nu(1-\beta_1)}{1-\beta_1-3\beta_2} I^2 + 2(5-\nu) I^1\right] \tag{3.10}$$

$$I^1_{\alpha\beta\lambda\mu} = \frac{1}{2}(\delta_{\alpha\lambda}\delta_{\beta\mu} + \delta_{\alpha\mu}\delta_{\beta\lambda}), \quad I^2_{\alpha\beta\lambda\mu} = \delta_{\alpha\beta}\delta_{\lambda\mu}$$

$$\beta_1 = \frac{32}{675} q \frac{(7-5\nu)(5-\nu)}{(2-\nu)}, \quad \beta_2 = \frac{32}{675} q \left(\frac{5-\nu}{2-\nu} + 10\nu\right)$$

where $q$ is the same as in (3.8).

The above formulas were derived on the assumption that all cracks expand in the external stress field. But this is not true for all external fields. A part of the cracks or even all of them may close, thus affecting the effective elastic constants. A method for allowing for cracks closed by the external field was considered in /11/.

Let us now consider the formula provided by the method of effective medium for the tensor of effective elastic constants. Equation (2.14) with allowance for (3.2) is of the form

$$B_* = B + \frac{2\pi}{3v_0}\langle a^3 P_*(a, b)\rangle \tag{3.11}$$

where $P_*$ is determined from the solution of the problem of an elliptic crack in a medium with the elasticity tensor $B_*$.

Let an isotropic medium contain a homogeneous set of circular cracks uniformly distributed with respect to orientation. The medium is assumed macroisentropic and $B_*$ defined by formula of the form

$$B_* = B + \frac{2}{3} q P_*, \quad P_* = \frac{4}{15}\frac{1-\nu_*}{\mu_*(2-\nu_*)}[\nu_* I^2 - 2(5-\nu_*) I^1]$$

where the quantities $q, I^1, I^2$ are those defined in (3.8) and (3.10).

From this, for the effective elastic constants $\mu_*$ and $\nu_*$ of a medium with cracks, we obtain formulas

$$\frac{\mu_*}{\mu} = 1 - \frac{32}{45} q \frac{(1-\nu)(5-\nu)}{(2-\nu)}, \quad q = \frac{45}{16}\frac{(\nu-\nu_*)(2-\nu_*)}{(1-\nu_*^2)[10\nu - \nu_*(1+3\nu)]} \tag{3.12}$$

of which the last may be considered as the equation for $\nu_*$. This result was obtained in /3/.

It was noted in /3/ that for $q \cong 9/16$ the shear modulus $\mu_*$ in (3.12) becomes negative, and these formulas loose any physical meaning. Note that $\mu_*$ and $\nu_*$ calculated by the method of effective field in the case of a Poisson set of cracks (3.9) are always positive. In the case of a model with restriction on crack intersections the components of $B_*$ (in (3.10)) vanish for $q \cong 2$ so that also here the region, where the method of effective field yield physically inconsistent results, is one and a half times wider. (For small $q$ both methods yield the same effective elastic constants).

One more point should be mentioned. The construction of tensor $P_*$ and the solution of Eq. (3.11) is technically extremely complicated in the case of arbitrary anisotropy of tensor $B_*$. Because of this, the method of effective medium is applied only when an inhomogeneous medium is macroisotropic. On the other hand, the method of effective field does not introduce any additional technical difficulties in the determination of $B_*$ when applied to a macroanisotropic medium.

4. Let us now consider the plane problem. The formalism of the method of effective field is applicable in the plane case without any fundamental alterations. Because of this, we present here only the final expressions for the effective elastic constants (the plane problem was considered in detail in /12/).

In the plane case a straight slit is the analog of the elliptic crack. For a Poisson set of straight cracks of random length $2l$ uniformly distributed with respect to orientations the medium is macroisotropic, and the Young's modulus $E_*$ and Poisson's ratio $\nu_*$ are of the form

$$\frac{E_*}{E} = \frac{\nu_*}{\nu} = \frac{1}{1+q}, \quad q = \frac{\pi\langle l^2\rangle}{\omega_0} \tag{4.1}$$

where $E$ and $\nu$ are the elastic constants of the original isotropic medium, and $\omega_0$ is the average area of a crack.

For an initially isotropic medium model with restriction on the intersection of cracks uniformly distributed with respect to orientation, the effective elastic constants are determined by formulas

$$\frac{E_*}{E} = \left[1 + \frac{q\,(1-3q/8)}{(1-q/2)(1-q/4)}\right]^{-1} \qquad (4.2)$$

$$\frac{\nu_*}{\nu} = \frac{E_*}{E}\left[1 - \frac{q^2}{8\nu\,(1-q/2)(1-q/4)}\right] \qquad (4.3)$$

The curves *1*, *2*, and *3* shown in Fig.1 represent, respectively, functions (4.1), (4.2) and (4.3). They are compared there with experimental data cited in /13/. Experiments were carried out on thin rubber sheets with a set of rectilinear through slits ($\nu = 0.5$). Experimental data are approximated by the dash line with small circles for $(E_*/E)$ and by dash-dot line for $(\nu_*/\nu)$.

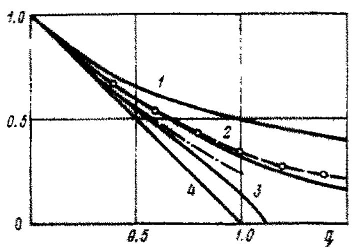

Fig.1

The statistical analysis shows that the set of cracks investigated in /13/ is satisfactorily defined by the model with restriction on crack intersections. The formulas for the effective elastic constants obtained for this model are in good agreement with experimental data.

Note that in this case the method of effective medium yields for $E_*$ and $\nu_*$ the following expressions:

$$E_*/E = \nu_*/\nu = 1 - q$$

to which the straight line *4* corresponds in Fig.1.

5. The method of effective field was used above for calculating the effective elastic constants of a medium with cracks. The possibilities of this method are, however, not limited to such cases. The method was applied in /14/ for constructing the first two statistical moments of the solution of the problem of constant electric current in a medium with a large number of cracks. Besides the $H_1$ hypothesis on the constancy of the effective field for each crack, the hypothesis of statistical independence of that field in region $\Omega_k$ from the orientation and size of the crack $\Omega_k$ (the $H_2$ hypothesis) was used there. It is possible to obtain on these assumptions closed equations for the second statistical moment of the effective field and, then using it, to determine the second statistical moment of solution. The second moments of the problem of the theory of elasticity considered here can be similarly obtained.

We stress that the hypotheses on which this method is based relate to the approximation of the state of each crack in an inhomogeneous medium. With increased crack concentration these hypotheses provide an increasingly rought description of the state of cracks. However, this may weakly affect the accuracy of determination of the first and second statistical moments of solution which are of the greatest interest in applications. These moments are in themselves fairly rough statistical characteristics with part of the information about details of crack behavior is certainly lost in them. Because of this an exact definition of the state of each crack in the calculation of the first two moments of the solution is hardly justified.

As shown by the results of the present investigations and of those in/4,12/, the method of effective field makes possible a satisfactory description of the results of experimental determination of the effective elastic constants and, also, provides a good correlation with the exact values of these constants in the case of the lattice of inclusions and cracks in a plane.

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